COLLECTION OF COMPRESSIBLE SEDIMENTS BY FILTRATION

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A complete mathematical model is proposed for the process of filtration separation of a suspension. The problem of collecting the sediment on a plane barrier in a regime of a specified pressure difference has been solved.

1. Let us examine the process of separating a suspension into the filtrate and the moist sediment by means of a filtration barrier. In the construction of the mathematical model of this process we will proceed from the following considerations: 1) the filtration resistance of the precipitate retards the process and makes it rather rather free of inertia, so that with considerable accuracy the pressure distribution within the suspension is hydrostatic; 2) the precipitation of the particles in the suspension proceeds in the effective direction of the mass forces; 3) the stress-strain state of the precipitate is described by the equations from the mechanics of saturated porous media; 4) filtration through the compressed precipitate is subject to the generalized Darcy law.

In actual practice, the filtration barrier is usually flat or cylindrical. In the former case, it is positioned horizontally, so that the filtration occurs under the action of the force of gravity and can be ascribed to a vertical pressure gradient. For purposes of intensifying the process, the pressure difference is increased either through vacuum rarefaction at the outlet from the filtration stream, or by means of a piston coming out of the suspension. In the case of a cylindrical filtration barrier, it is centrifugal force that plays the fundamental role, and the angular rotational velocity ω of the barrier is so great that the force of gravity can be neglected, in which connection the filtration flow is virtually plane-radial.

Either case can be uniquely described in the cylindrical coordinate system (r, θ, z). In the region D_1 occupied by the suspension we have [1]

$$[c\rho_s + (1-c)\rho]a - \frac{\partial p}{\partial x} = 0, \tag{1}$$

$$c\rho_s a - c\frac{\partial p}{\partial x} - f\left(\frac{q_s}{c} - \frac{q}{1-c}\right) = 0, \tag{2}$$

$$x^n\frac{\partial c}{\partial t} + \frac{\partial (x^n q_s)}{\partial x} = 0, \tag{3}$$

$$x^n\frac{\partial (1-c)}{\partial t} + \frac{\partial (x^n q)}{\partial x} = 0, \tag{4}$$

where for the horizontal grid we have $x \equiv z$, $n = 0$, $a = g\cos\alpha$, while for the cylindrical grid we have $x \equiv r$, $n = 1$, $a = \omega^2 r$. The x axis is directed to coincide with the direction of the flow.

In region D_2 occupied by the moist precipitate, in accordance with the theory of saturated porous media [2]:

$$[(1-m)\rho_s + m\rho]a - \frac{\partial p}{\partial x} + \frac{1}{x^n}\frac{\partial (x^n\sigma_{xx})}{dx} = 0, \tag{5}$$

$$\rho a - \frac{\partial p}{\partial x} - \frac{1}{K}\left(W - \frac{m}{1-m}W^s\right) = 0, \tag{6}$$

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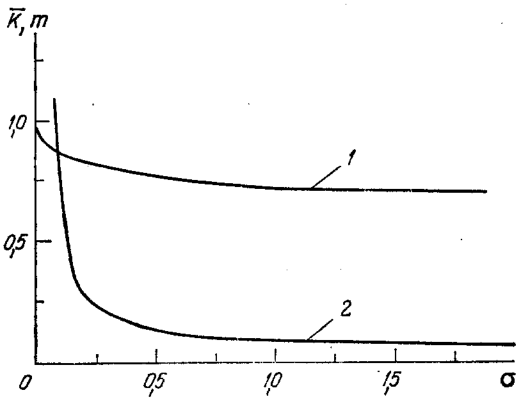

Fig. 1. Porosity m (1) and dimensionless permeability $\bar{K}$ (2) as functions of the dimensionless stress σ.

$$x^n \frac{\partial m}{\partial t} + \frac{\partial (x^n W)}{\partial x} = 0, \tag{7}$$

$$x^n \frac{\partial (1-m)}{\partial t} + \frac{\partial (x^n W^s)}{\partial x} = 0. \tag{8}$$

In order to close system of equations (1)-(8) we have to specify the relationships f(c), m(σ), and K(σ). These are found experimentally (see Fig. 1). The law governing the growth in the thickness of the precipitate layer h(t) is determined by the conditions of mass balance for the liquid and solid phases at the boundary of conjugacy between the regions D_1 and D_2, $x = x_0(t)$. Obviously, $dx_0/dt = -dh/dt$. From the general conditions at surfaces of strong discontinuities [3] it follows directly that

$$\frac{dh}{dt} = \frac{W-q}{1-c-m} = \frac{q_s - W^s}{1-c-m}, \quad x = x_0. \tag{9}$$

In specific problems Eqs. (1)-(9) are enhanced by the conditions at the inlet to the flow of the suspension and at the filtration barrier. In the construction of the solutions it is useful to take into consideration the independence from x of the total volumetric flow rate of the incompressible phases which follows out of (1), (2), and (7), and (8):

$$Q(t) = x^n (q + q_s) = x^n (W + W^s). \tag{10}$$

We must also bear in mind that $\sigma(x_0) = 0$ [2].

2. Let us examine the assumptions usually made in calculating the precipitate collection, and on the basis of (1)-(9) we will ascertain the conditions under which they are applicable.

Equation (5) is not to be found in the literature, with which we are familiar, dealing with the filtration separation of suspensions; in the place of this equation use is made of the linear relationship between σ and p, or it is assumed that K = K(p), m = m(p) (see, for example, [4, 5]). For flat barriers, with $\rho gh \ll p_0 - p_a$, such an approach is completely valid, since in this case we can neglect the mass forces in (5), and then

$$\sigma = (p_0 - p)/p_a, \quad p_0 = p(x_0, t). \tag{11}$$

Conversely, in the case of centrifuging, because of (5), we have

$$\int_{x_0}^{x} \frac{d(x\sigma)}{x} = \frac{p_0 - p}{p_a} + \frac{\omega^2}{p_a} \int_{x_0}^{x} [(1-m)\rho_s + m\rho]\, x dx,$$
$$x_0 \leqslant x \leqslant R, \quad R = x_0 + h. \tag{12}$$

Simple estimates show that in real regimes ($h \ll R$, $\omega \gtrsim 10^3$/sec) the fundamental contribution to the right-hand portion of (12) is made by the second term. In this case

$$\int_0^{\sigma} \frac{d\sigma}{\rho_s + (\rho - \rho_s) m} = \frac{\omega^2}{2p_a} (x^2 - x_0^2).$$

Thus, in the case of centrifuging we cannot use formula (11), as is done, for example, in [4].

The solution of these problems is substantially simplified, if we assume the function x^nW to be independent of x. In this connection, of interest is the bilateral estimate of its change in the region $D_2(x_0 \le x \le x_1)$. We will denote $W_0 = W(x_0, t)$, $W_1 = W(x_1, t)$, $\lambda = (x_1^nW_1)/(x_0^nW_0)$. It is clear from physical considerations that $\partial m/\partial t < 0$, so that in the light of (7), x^nW increases monotonically with a rise in x, $\min x^nW = x_0^nW_0$, $\max x^nW = x_1^nW_1$, and, consequently, $\lambda > 1$.

We can obtain the upper bound of λ in two ways. The first of these is based on the fact that the filtration rate is positive when $x = x_0$: $W_0 - W_0^sm_0/(-m_0 + 1) > 0$. In view of (10), when $W^s(x_1) = 0$, we have $W_0^s = (x_1/x_0)^nW_1 - W_0$, which means that $(1 - m_0)W - m_0[(x_1/x_0)^nW_1 - W_0] > 0$, i.e., $\lambda < (1/m_0)$.

In the second case, we make use of the physically obvious inequality $(dh/dt) > 0$ and Eqs. (1), (2), (9), and (10). On the basis of (9) and (10),

$$\lambda = \frac{Q(t)}{x_0^nW_0} < \left.\frac{q+q_s}{q}\right|_{x=x_0}.$$

The ratio q_s/q is found as a result of the elimination of $(\partial p/\partial x)$ from (1) and (2):

$$\frac{q_s}{q} = \frac{c}{1-c} + c^2(1-c)\frac{(\rho_s-\rho)a}{fq}. \tag{13}$$

Thus, the sought estimates are of the form

$$1 < \lambda < (1-m_0);\ 1 < \lambda < \frac{1}{1-c_0} + c_0^2(1-c_0)\frac{(\rho_s-\rho)a}{fq_0},$$
$$c_0 = c(x_0,\ t),\ q_0 = q(x_0,\ t).$$

Consequently, the assumption to the effect that x^nW is independent of x has been validated for values of $m_0 = m(x_0t)$, close to unity (fibrous suspensions), as well as with a small volumetric concentration of particles in the suspension. Here, accurate to εQ, it is natural to assume that $x^nW(x, t) \simeq Q(t)$, with the small parameter ε denoting c_0 or $1 - m_0$. We will stress that in this approximation the difference $W_0 - q_0$ which figures in (9) must be expressed with an accuracy higher than εQ, since otherwise the ratio $(W_0 - q_0)/(1 - c_0 - m_0)$, and together with it, Eq. (9) for the increase in sedimentation will provide no information.

3. Let us examine the collection of the precipitate on a plane horizontal barrier when $W(x, t) \simeq Q(t)$ in the regime of the following specified pressure difference:

$$p(x_0,\ t) = p_0,\ p(x_1,\ t) = p_1. \tag{14}$$

Let $h(0) = 0$, $\rho gh(t) \ll (p_0 - p_a)$. The filtration equation (6) can then be written approximately in the following form:

$$Q(t) \simeq -K_0\overline{K}(\sigma)\frac{\partial p}{\partial x}, \tag{15}$$

where $\bar{K}(\sigma) = K(\sigma)/K_0$ is the dimensionless permeability. Having integrated (15) over the thickness of the precipitate, with consideration of (11), we find

$$Q(t)h(t) = -\int_{x_0}^{x_1} K(\sigma)\frac{\partial p}{\partial x}dx = p_aK_0\int_0^{(p_0-p_1)/p_a}\overline{K}(\sigma)d\sigma = A. \tag{16}$$

Now, from (15) and (16) we have

$$dx = \frac{p_aK_0}{Q(t)}\overline{K}(\sigma)d\sigma = \frac{h}{A}p_aK_0\overline{K}(\sigma)d\sigma,$$

so that consequently:

$$\int_{x_0}^{x_1} mdx = \frac{hp_aK_0}{A}\int_0^{(p_0-p_1)/p_a} m(\sigma)\overline{K}(\sigma)d\sigma = \frac{hB}{A} = h\langle m\rangle. \tag{17}$$

Equations (16) and (17) show that in the approximation ($W \approx Q$) with which we are dealing here, when $p_0 - p_1$ = const, the product Qh and the average porosity $\langle m \rangle$ for the porosity of the precipitate all remain constant (i.e., they are conserved) in the process of suspension separation and they are all determined only by the pressure difference and the material functions $\bar{K}(\sigma)$ and $m(\sigma)$.

Before we make use of Eq. (9) to find h(t), with sufficient accuracy we will express $(W_0 - q_0)$ in terms of Q(t), h(t), and c_0. From (10) and (13) we find

$$q_0 = (1 - c_0) Q - \varphi(c_0), \ \varphi(c) = \frac{c^2 (1-c)^2 (\rho_s - \rho) a}{f(c)}.$$

Having integrated (7) over the thickness of the precipitate, we have

$$\int_{x_0}^{x_1} \frac{\partial m}{\partial t} dx + Q(t) - W_0 = 0.$$

Since

$$\int_{x_0}^{x_1} \frac{\partial m}{\partial t} dx \equiv \frac{d}{dt} \int_{x_0}^{x_1} m dx + m_0 \frac{dx_0}{dt},$$

so that with consideration of (7)

$$W_0 = Q(t) + (\langle m \rangle - m_0) \frac{dh}{dt}.$$

Thus:

$$W_0 - q_0 = c_0 Q(t) + \varphi(c_0) + (\langle m \rangle - m_0) \frac{dh}{dt}$$

and on the basis of (9) and (16) we have

$$(1 - c_0 - \langle m \rangle) \frac{dh}{dt} = \frac{c_0 A}{h} + \varphi(c_0). \tag{18}$$

Let us note that when $\psi(c_0) < 0$ (the force of gravity is directed against the flow) the right-hand side of (18) may reduce to zero with the passage of time, and the collection of the precipitate then becomes impossible. If the quantity c_0 is independent of time, then as a result of the integration of (18) we obtain

$$t = \frac{1 - c_0 - \langle m \rangle}{\varphi(c_0)} \left[h - \frac{c_0 A}{\varphi(c_0)} \ln \left(1 + \frac{\varphi(c_0)}{c_0 A} h \right) \right].$$

Since $1 + \varphi(c_0)h/(c_0 A) > 0$, with $\varphi(c_0) < 0$ the following inequality is valid:

$$h(t) < \frac{c_0 A}{|\varphi(c_0)|}.$$

In the particular case $\varphi(c_0) = 0$ (the suspension exhibits either $f = \infty$ or $\rho_S = \rho$) the solution of (18) has the form

$$h = \sqrt{\frac{2 c_0 A}{1 - c_0 - \langle m \rangle} t}.$$

When $m(\sigma) = m_0$ = const, $K(\sigma) = K_0$ = const ($\langle m \rangle = m_0$, $A = p_a K_0$) it changes into the one familiar from [6, 7].

Strictly speaking, concentration $c(x_0)$ at the free boundary of the precipitate is independent of time only for suspensions. In the remaining cases, as a consequence of the precipitation of particles within the suspension, it is variable and determined by Eqs. (1)-(4) and the function Q(t). From (10) and (13) we find that $q = (1 - c)Q - \varphi(c)$. Then, in the light of (4)

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x} (cQ + \varphi(c)) = 0, \ x_*(t) \leqslant x \leqslant x_0(t).$$

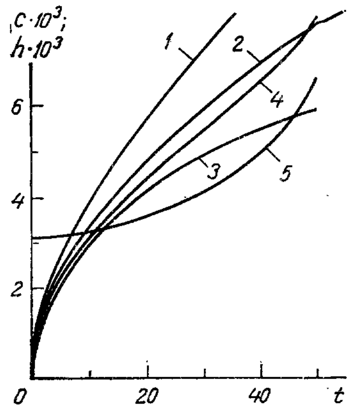

Fig. 2. Volumetric concentration c and precipitate thickness h as functions of time t: 1) h(t), $c = 0.31 \cdot 10^{-2}$, $\alpha = 0$; 2) h(t), $c = 0.31 \cdot 10^{-2}$, $\alpha = \pi/2$; 3) h(t), $c = 0.31 \cdot 10^{-2}$, $\alpha = \pi$; 4) h(t), $c(t) = c$, $\alpha = \pi$; 5) c(t), $\alpha = \pi$. h, m; t, sec.

In actual fact, the specified pressure $p(x_*) = p_0$ is applied to the plane $x = x_*(t)$. The transfer of this value to the boundary of the precipitate $x = x_0(t)$ [condition (14)] is justified by the fact that in actual practice $\rho g(x_0 - x_*) \ll p_0$.

When $x = x_*$ the true velocities of the particles in the liquid and solid phases coincide with dx_*/dt: $q/(1 - c) = q_s/c = \dot{x}_*$, and therefore $q = q_s = Q(t) = \dot{x}_*$.

Let us introduce into our consideration the reckoning system (ζ, t), moving toward the filtration barrier at the velocity $Q(t)$: $\zeta = x - x_*(t)$. In these new variables we have the following problem for the determination of $c(\zeta, t)$ in D_1:

$$\frac{\partial c}{\partial t} + \frac{\partial \varphi}{\partial c}\frac{\partial c}{\partial \zeta} = 0, \; c(\zeta, 0) = \hat{c}(\zeta), \; 0 \leqslant \zeta \leqslant x_0 - x_*(t).$$

From a mathematical standpoint this is entirely analogous to the problem of the gravitational separation of water and petroleum, which has been so thoroughly studied in the theory of two-phase filtration [8, 9]. Its solution determines the value of the concentration $c_0 = c(x_0 - x_*, t)$ at the free boundary of the precipitate in the general case.

For fibrous suspensions the effect of precipitation makes itself evident only in the limited region of zero concentration values, so that it is therefore permissible to regard the change in $c(\zeta, t)$ in D_1 in the integral sense, making the assumption that

$$c_0(t) \equiv c(x_0 - x_*, t) \simeq \frac{1}{x_0 - x_*} \int_{x_*}^{x_0} c(x, t)\,dx.$$

In this case the function $c_0(t)$ has the ratio of the volume $V_s(t)$ of the solid particles to the overall volume $V(t)$ of the suspension in D_1 is expressed as follows:

$$c_0(t) = \frac{V_s(0) - S \int_{x_0}^{x_1} (1 - m)\,dx}{V(0) - Sh - S \int_0^t Q(\tau)\,d\tau}.$$

Hence

$$\frac{V(0)}{S} - h - \int_0^t Q(\tau)\,d\tau = \left(\frac{1}{c_0}\right)[(V_s(0)/S) - h(1 - \langle m \rangle)].$$

Differentiation of this relationship over time, with consideration of (16) and (18), leads to the equation

$$\frac{dc_0}{dt} = -\frac{c_0 \varphi(c_0)}{(V_s(0)/S) - (1 - \langle m \rangle)h}. \tag{19}$$

Figure 2 shows the results from the calculations carried out with Eqs. (18) and (19) for a water suspension whose solid phase is made up of 30% fibrous (a 40° Schopper–Riegler grinding ratio) and 70% powder (particle dimensions of $2 \cdot 10^{-5}$–$5 \cdot 10^{-5}$ m) cellulose, with

$V(0) = 10^{-3}$ m^3, $S = 0.2$ m^2, $c_0(0) = 0.31\cdot10^{-2}$, $p_0 - p_1 = 2p_a$, $K_0 = 10^{-9}$ m^2/(Pa·sec). The functions $K(\sigma)$ and $m(\sigma)$, found experimentally (Fig. 1), correspond to the constant values $A = 3.62\cdot10^{-5}$ m^2/sec, $B = 2.93\cdot10^{-5}$ m^2/sec which figure in (16) and (17).

The function $f(c)$, based on experimental data, was approximated in the range of changes in the concentration, i.e., $2\cdot10^{-3} \le c \le 1.5\cdot10^{-2}$, in the following form:

$$f(c) = 0.185\cdot10^{-3}\cdot c^{3.375}\ \text{kg/m}^3\cdot\text{sec}.$$

The calculation results (Fig. 2) are in good agreement with experiment.

NOTATION

c, volumetric concentration of the particles in the suspension; ρ_s and ρ, true densities of the solid particles and of the liquid; p, pressure; p_a, atmosphereic pressure; g, gravitational acceleration; α, angle of inclination for the force of gravity to the direction of the flow; ω, angular velocity of the barrier; x, generalized coordinate; x_0 and x_1, coordinates of the precipitate boundaries; x_*, coordinate of the suspension boundary; q and q_s, volumetric flow rates of the liquid and solid phases in the suspension, per unit area; W and W^s, volumetric flow rates of the liquid and solid phases in the precipitate per unit area; σ_{xx}, effective stress; K, filtration factor (permeability); $\bar{K} = K/K_0$, dimensionless permeability; m, porosity of precipitate; h, thickness of precipitate; t, time; R, radius of cylindrical barrier; ε, small parameter; V_s, volume of solid particles in suspension; V, volume of suspension; S, area of filtration barrier. The subscripts 0 and 1 indicate the corresponding quantities have been taken at the boundary between the suspension and the precipitate and the filtration barrier.

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